Affine Grassmanians for GL_n

Notation:

- R a ring, $n \in \mathbb{Z}_{\geq 1}$.
- $R[[t]]$ ring of formal power series
  $\cap$
- $R((t)) := R[[t]][\frac{1}{t}]$ overring of Laurent series.

Def: A $R[[t]]$-lattice in $R((t))^n$ is a finite loc. free $R[[t]]$-mod. $\Lambda \subset R((t))^n$ s.t. $\Lambda[t^{-1}] = R((t))^n$.

$\Lambda_0 := \mathbb{Z}[[t]]^n$, $\Lambda_{0,R} := R[[t]]^n$

Def: Affine Grassmanian: for $G = GL_n$ is a presheaf of sets:

$Gr: \text{Rings} \longrightarrow \text{Set}$

$Gr(R) := \{ \Lambda_{0,R}\text{-lattices in } R((t))^n \}$

$Gr(R) =$ "R-family of lattices"

$GL_n(R((t))) \circlearrowright Gr(R)$ via $(g, \Lambda) \mapsto g\Lambda$

Aim: put algebro-geometric structure on Gr.

Def: A <u>strict ind-scheme</u> is a functor

$X : \mathrm{Affsch}^{op} = \mathrm{Rings} \longrightarrow \mathrm{Set}$ s.t.

$X \simeq \operatorname{colim}_{i \in I} X_i$ where $(X_i)_i$ filtered. syst. of schem

w/ transition maps $X_i \longrightarrow X_j$, $j \geq i$ closed immersions.

So just think "Ind-scheme" = union of schemes.

$\rightsquigarrow$ categ. of IndSch, obj = ind-schemes

morph = maps of functors.

If I is countable, then $\rightsquigarrow$ "strict $\aleph_0$-ind-schemes".

Exple: i) $\mathrm{Sch} \underset{\substack{\text{embedding}\\ \text{by Yoneda}}}{\overset{\text{full}}{\hookrightarrow}} \mathrm{Ind\,Sch}$

ii) $\mathbb{A}^\infty := \bigcup_{i \geq 0} \mathbb{A}^i$ w/ $\mathbb{A}^i \subset \mathbb{A}^{i+1}$ via the

first i coordinates.

iii) I ind-Set

$\rightsquigarrow$ $\mathbb{A}_{\mathbb{Z}, I} = \operatorname{colim}_{\substack{J \subset I \\ \text{finite}}} \mathbb{A}_{\mathbb{Z}}^{|J|}$ represents the functor

$$R \longrightarrow \bigoplus_{i \in I} R$$

$\rightsquigarrow$ $\mathbb{P}_{\mathbb{Z}, I} = \operatorname{colim}_{\substack{J \subset I \\ \text{finite}}} \mathbb{P}_{\mathbb{Z}}^{|J|}$ is an ind-scheme

Lemma: T : scheme, $(T_j \to T)_j$ a fpqc cover, X ind-scheme then

$$\mathrm{Hom}(T, X) \to \prod_i \mathrm{Hom}(T_i, X) \rightrightarrows \prod_{i,j} \mathrm{Hom}(T_i \times_T T_j, X)$$ is exact.

Proof: $T = \operatorname{colim}_{\substack{U \subseteq T \\ \text{open q-compact}}} U$ even if T is not qc.

WLOG: T qc, $(T_j \to T)_j$ finite cover.

T_j affine $\Rightarrow$ single cov. $T' = \bigsqcup_j T_j \twoheadrightarrow T$ where both T, T' are qc.

Fact: $\mathrm{Hom}(T, X) = \operatorname{colim} \mathrm{Hom}(T, X_i)$
$\uparrow$
qc $\quad X = \operatorname{colim} X_i$

WLOG X is a scheme. But schemes are fpqc-sheaves.

Cor: X ind-sch $\Rightarrow$ X fpqc sheaf in AffSch

Lemma: Let $X \to Y$ of functors, $\mathrm{AffSch}^{op} \to \mathrm{Sets}$. TFAE:

i) $\forall$ aff sch $T \to Y$, $X \times_Y T$ scheme

ii) $\forall$ schemes $T \to Y$, $X \times_Y T$ scheme $\quad \boxed{/\!/}$

In this case, we say that $X \times_Y T$ is representable by schemes.

E.g.: $X = \operatorname{colim}_i X_i$ ind-scheme $\xRightarrow{\forall i \in I}$ $X_i \hookrightarrow X$ is representable by closed immersion.

Prop: IndSch has following properties:

i) final object is $\operatorname{Spec}\mathbb{Z}$,
ii) closed under fibre products $\Big\} \Rightarrow \exists$ finite limits (Tag [002O])

iii) closed under filtered limits (w/ aff. transition maps).

iv) " " disjoint unions.

Def: X ind-scheme.

$$\underline{\text{top. space}} \quad |X| := \operatorname*{colim}_{k \text{ field}} X(k)$$

equipped with topology given by subfunctors which are representable by closed immersions.

E.g.: • X scheme $\rightsquigarrow$ $|X|$ is usual top space.

• we can check:

$$X = \operatorname*{colim}_i X_i \rightsquigarrow \operatorname*{colim}_i |X_i| \xrightarrow{\text{homeo}} |X|$$

$|X|$ is hence independent from the choice of presentations.

Def: i) Let P be a local property of schemes

An ind-sch. is called P if $\exists$ $X = \operatorname{colim}_i X_i$ s.t. each X_i is P

ii) Let P be a property of of maps of schemes that is local on the target (e.g. affine, proper, l.f.t., closed immersion etc ... Then a morph of ind-schemes $f: X \to Y$ has ind-P if $\exists$ $X = \operatorname{colim} X_i$, $Y = \operatorname{colim} Y_j$,

$f = \lim_j \operatorname{colim}_i f_{ij}$, $f_{ij}: X_i \to Y_j$ if each f_{ij} has P

E.g.:

- $X = \operatorname{colim} X_i$ ind-scheme, I ind-set

  $\rightsquigarrow$ $\mathbb{A}_{\mathbb{Z}, I} \longrightarrow \operatorname{Spec} \mathbb{Z}$ is ind-affine and ind-smooth

  $\rightsquigarrow$ $\mathbb{P}_{\mathbb{Z}, I} \longrightarrow \operatorname{Spec} \mathbb{Z}$ is ind-proper

- X qcqs formal scheme = X qcqs ind-scheme s.t. X_{red} is a scheme.

Base change: Let S be a scheme.

$\rightsquigarrow$ slice cat. of obj IndSch_S

$=$ full subcat func $X : \mathrm{AffSch}_S^{op} \longrightarrow \mathrm{Set}$ s.t.

$X = \operatorname{colim}_i X_i$ by S-schemes

Back to affine Grassmanians:

$\forall\, a, b \in \mathbb{Z}$, $a \leq b$,

$$\mathrm{Gr}_{[a,b]}(R) := \{ \Lambda \in \mathrm{Gr}(R) \mid t^b \Lambda_{0,R} \subset \Lambda \subset t^a \Lambda_{0,R} \}$$

$\rightsquigarrow$ filtered system of subfunctors

$\rightsquigarrow$ Gr is exhaustive and $\mathrm{Gr} = \operatorname{colim}_{a \leq b} \mathrm{Gr}_{[a,b]}$

Thm: Each $Gr_{[a,b]} \longrightarrow \operatorname{Spec} \mathbb{Z}$ is representable by a proper morphism.

Proof: $\forall$ M a $\mathbb{Z}$-module,

$$\operatorname{Grass}(M)(R) := \{ N \subset M \otimes_{\mathbb{Z}} R \mid M \otimes_{\mathbb{Z}} R / N \text{ is finite loc. free} \}$$

Grass (M) is representable by a smooth proper scheme over $\mathbb{Z}$.

Fix $a \leq b$. , $M_{[a,b]} := t^a \mathbb{Z}[[t]]^m / t^b \mathbb{Z}[[t]]^m \cong \mathbb{Z}^{m(b-a)}$

steps: ① : $Gr_{[a,b]} \longrightarrow \operatorname{Grass}(M_{[a,b]})$

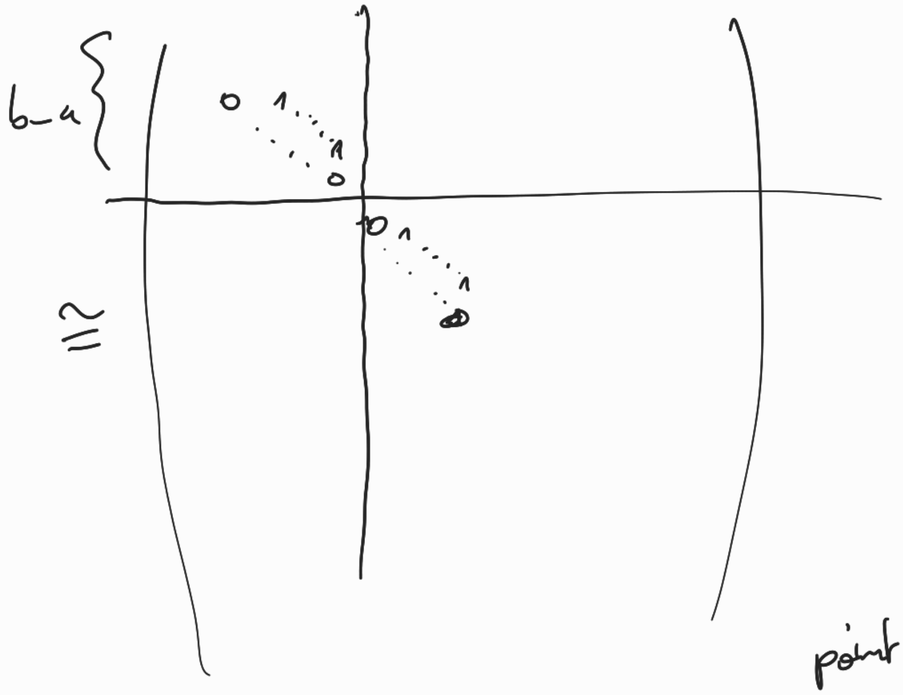
$$\Lambda \longmapsto \Lambda / t^b \Lambda_{0,R}$$

check well-defined.

② : $\operatorname{Grass}^t_{[a,b]}(M) \underset{\substack{\text{closed}\\ \text{subfunc}}}{\subset} \operatorname{Grass}(M_{[a,b]})(M) = \coprod_{r \geq 0} \operatorname{Grass}(r, M)$

① + ② $\Longrightarrow$ Theorem.

For ② $t \curvearrowright M$ $\mathbb{Z}$-linear nilp. operator. So if $(e_1, \dots, e_m)$ is the std $\mathbb{Z}[t]$-basis of Λ_0 then a basis of M is $(t^a e_1, \dots, t^{b-1}, t^a e_2 \dots)$ so the operator is given by Jordan blocs



points in the classical Grassm. which are t-stable

$$\Longrightarrow \mathrm{Grass}(M_{[a,b]})(R) = \left\{ N \in \mathrm{Grass}(M_{[a,b]}(R)) \,\middle|\, tN \subset N \right\}$$

$$=: \mathrm{Grass}^{t}(M_{[a,b]})(R)$$

So $Gr_{[a,b]} \longrightarrow \mathrm{Grass}(M_{[a,b]})$ is closed subfunctor.

For 1) <u>well defined</u>

: localizing on $R \overset{\text{Exercise}}{\Longrightarrow} \Lambda$ free $R[[t]]$-mod

So $R((t))^n/\Lambda = \Lambda[t^{-1}]/\Lambda \overset{\cong}{\longleftarrow} \bigoplus_{i \geq 1} t^{-i} R^n$

So $R((t))^m / \Lambda$ is free R-mod
$\cup$
so $t^a \Lambda_{0,R} / \Lambda$ is finite loc. free R-mod

$\rightsquigarrow$ $Gr_{[a,b]} \hookrightarrow Grass^t(M_{[a,b]})$ mono.

<u>Surjective:</u> we already know that

$Grass^t(M_{[a,b]})$ is of finite type $/ \mathbb{Z}$.

WLOG
$\rightsquigarrow$ R f.g. R-alg.

Take $N \in Grass^t(M_{[a,b]})(R)$.

$\rightsquigarrow \Lambda := \ker\left(t^a \Lambda_{0,R} \longrightarrow t^a \Lambda_{0,R} / t^b \Lambda_{0,R} = M_{[a,b]} \otimes_{\mathbb{Z}} R \longrightarrow (M_{[a,b]} \otimes_{\mathbb{Z}} R)/N\right)$

$\rightsquigarrow \Lambda_f := \ker\left(t^a R[t]^m \longrightarrow t^a \quad\rule{8cm}{0.4pt}\quad\right)$

R noetherian $\implies$ $R[t] \longrightarrow R[[t]] = \widehat{R[t]}$ is flat

$\left(M \longrightarrow M \otimes_A \hat{A} \text{ exact in the cat of f.g. } A\text{-modules for } A \text{ noeth. ring} \right)$

flatness
$\rightsquigarrow$ $\Lambda_f \otimes_{R[t]} R[[t]] = \Lambda$

Λ_f is finite R-flat

enough
$\rightsquigarrow$ (Nakayama): $\forall\, m \in \max Spec(R)$: $\Lambda_f \otimes_R R/m$ is finite loc. free

But $\Lambda_f \otimes R/m$ is a torsion-free $R/m[t]$-submod of $t^a R/m[t]^m$

But $R/m[t]$ is a PID so $\Lambda_f \otimes_R R/m$ is finite free ▨

Cor:

$$\left\{ \begin{array}{l} \text{fin. loc. free } R[t]\text{-submod} \\ \Lambda_b \in R[t,t^{-1}]^n \text{ s.t.} \\ \Lambda_b[t^{-1}] = R[t,t^{-1}]^n \end{array} \right\} \xrightarrow{\cong} Gr(R)$$

$$\Lambda_b \longmapsto \Lambda_b \otimes_{R[t]} R[[t]]$$

$$\alpha^{-1}\Gamma(\mathbb{A}^1_R, \mathcal{E}) \subset R[t,t^{-1}]^n$$

$$\uparrow$$

$$(\mathcal{E}, \alpha)$$

$$\|$$

$$\left\{ (\mathcal{E}, \alpha) \;\middle|\; \begin{array}{l} \mathcal{E} \text{ is v.B. on } \mathbb{A}^1_R \\ \alpha : \mathcal{O}^n_{\mathbb{A}^1_R \setminus \{0\}} \longrightarrow \mathcal{E}|_{\mathbb{A}^1_R \setminus \{0\}} \end{array} \right\} \Big/ \text{iso}$$

Cor: $Gr \longrightarrow \operatorname{Spec} \mathbb{Z}$ is ind-proper ind-scheme.

Proof:

$$\begin{tikzcd} Gr_{[a,b]} \arrow[rr, hook, "\text{proper + mono}", "j"'] \arrow[dr, "\text{proper}"'] & & Gr_{[a',b']} \arrow[dl, "\text{proper}"] \\ & \operatorname{Spec} \mathbb{Z} & \end{tikzcd}$$

$a' \leq a \leq b \leq b'$

So j is closed immersion (tag [04XV])

$\rightsquigarrow$ $Gr \in \mathrm{IndSch}$ and Gr ind-proper

Cor: • Gr is an fpqc sheaf

• Gr commutes w/ filtered colims (since Gr is union of schemes of finite type)

e.g. $Gr_n^{aff,[0,1]}$: $k[[t]]^n \supseteq \Lambda \supseteq t\,k[[t]]^n$

$$Gr_n^{aff,[0,1]} = \coprod_{d=0}^{n} Gr(d,n)$$

– Fix G a linear alg. reductive grp over k.

Def: the loop group LG of G is the functor

$$\begin{aligned} k\text{-alg} &\longrightarrow Set \\ R &\longmapsto LG(R) := G(R((t))) \end{aligned}$$

the positive loop group L^+G of G is the functor

$$\begin{aligned} k\text{-alg} &\longrightarrow Set \\ R &\longmapsto L^+G(R) := G(R[[t]]) \end{aligned}$$

E.g.: Let $G := GL_m$

$$L^+G = GL_m(R[[t]])$$

via $GL_m \longrightarrow \mathrm{Mat}_{m\times m} \times \mathrm{Mat}_{m\times m}$

$$A \mapsto (A, A^{-1})$$

Identify $L^+G := GL_m(R[[t]])$ with

$$\{(A,B) \in \mathrm{Mat}_{m\times m}(R[[t]]) \times \mathrm{Mat}_{m\times m}(R[[t]]) \mid AB = 1\}$$

So L^+G is closed subscheme of $\prod_{i\geq 0} \mathbb{A}^{m^2} \times \mathbb{A}^{m^2}$

hence is ∞-dim. scheme.

Def: the affine grassmanian for G

$$Gr'_G := LG / L^+G$$

(I) $G = GL_m$,

we describe Gr' in explicately in terms of lattices:

Claim: $Gr'_{GL_m} \cong Gr_{GL_m}$

Proof idea:

show $\rightsquigarrow$ $\forall\, \mathcal{L}$ a lattice, $\mathcal{L}$ is free over $R[[t]]$ fpqc-locally on R.

- choose a basis of $\mathcal{L}$ gives a representation by a matrix in $GL_n(R((t)))$, well def. up to an element of $GL_n(R[[t]])$.

enough this

- achieve this up to a fpqc-base change $R \longrightarrow R'$.